# On SCC-Recursiveness in Quantitative Argumentation

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**Abstract.** *Abstract argumentation* is a reasoning model for evaluating arguments based on various *semantics*. *SCC-recursiveness* is a sophisticated property of semantics that provides a general schema for characterizing semantics through the decomposition along *strongly connected components* (SCCs). While this property has been extensively explored in various qualitative frameworks, it has been relatively neglected in quantitative argumentation. To fill this gap, we demonstrate that this property is well-suited to *fuzzy extension semantics*, which is a quantitative generalization of classical semantics in *fuzzy argumentation frameworks* (FAF). We tailor the SCC-recursive schema to enable the characterization of fuzzy extension semantics through the recursive decomposition of FAF along its SCCs. Our contributions are twofold. Theoretically, we show that SCC-recursiveness provides an alternative approach to characterize fuzzy extension semantics, offering a deep understanding and better insight into these semantics. Practically, our schema provides a sound and complete algorithm for computing fuzzy extension semantics, which naturally reduces computational efforts when dealing with a large number of SCCs.

**Keywords:** Abstract Argumentation · Quantitative Argumentation · SCC-recursiveness

## 1 Introduction

Argumentation serves as a process for reasoning and decision-making in conflict situations, garnering significant attention in the field of Artificial Intelligence [11, 27]. Its applications span diverse domains, including reasoning with inconsistent information [29], decision making [3], explainable AI [37], etc.

Dung's seminal work on *argumentation framework* [17] (AF) is a well-studied formalism in argumentation theory. It abstracts argumentation scenarios as a directed graph whose nodes represent arguments and arrows represent attacks among arguments. In recent years, Dung's AF has been extensively explored in quantitative settings, leading to significant developments such as Fuzzy AF [15, 25, 34], Probabilistic AF [23, 26] and Weighted AF [12, 18]. These quantitative frameworks enrich the expressive power of classical AF by associating numerical values with arguments or attacks to capture uncertain information.


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The evaluation of arguments is a central topic in argumentation literature, commonly achieved through various *semantics* [4]. For instance, the well-known *extension semantics* [17] are proposed for Dung's AF to identify sets of jointly accepted arguments, while *gradual semantics* [1,15,32] are developed for quantitative frameworks to calculate the *acceptability degree* of arguments. Moreover, investigating the properties of semantics is crucial for their understanding, comparison and computation [2,6,9,30].

*SCC-recursiveness* [7] is a sophisticated property of semantics that relies on the graph-theoretical notion of *strongly connected components* (SCCs). Its significance lies in providing a general schema for characterizing semantics through the recursive decomposition of AF along its SCCs. Research on this schema has attracted extensive interest in the literature. First, it has proven to be one of the most efficient methods to reduce computational efforts [8,13,14,21]. Second, it has become a widely recognized property in principle analysis [20,30,31,35]. Third, many new semantics based on this schema have been proposed to address problematic behavior [7,19]. Finally, it has been extended to many other frameworks, such as *ADF* [22], *SETAF* [20], and *Unrestricted AF* [10].

Despite substantial contributions on this topic, almost all existing research on SCC-recursiveness has been restricted to qualitative settings. In contrast, research in various quantitative settings, such as probabilistic, fuzzy or weighted, remains *open* for investigation [28]. This limitation significantly restricts the applicability of SCC-recursiveness. It raises the question of how to define an SCC-recursive scheme to characterize semantics in quantitative settings. One underlying challenge is that, in these frameworks, arguments are often evaluated based on the degree to which they can be accepted, which seems incompatible with the schema.

In this paper, we show that SCC-recursiveness is well-suited to *fuzzy extension semantics*, introduced in [34] to generalize classical extension semantics within *fuzzy argumentation frameworks* (FAF). We tailor the existing SCC-recursive schema to enable the characterization of fuzzy extension semantics—including *admissible*[1], *complete*, *grounded* and *preferred*—through the recursive decomposition of FAF along its SCCs. Our contributions are twofold. Theoretically, we show that SCC-recursiveness provides an alternative approach to characterize fuzzy extension semantics, offering a deep understanding and better insight into these semantics. Our approach also paves the way for exploring SCC-recursiveness in other quantitative frameworks. Practically, our schema provides a sound and complete algorithm for computing fuzzy extension semantics, underpinned by several key theorems. As illustrated by examples, this algorithm naturally reduces computational efforts when dealing with a large number of SCCs, laying an implementation foundation for real-world applications.

The remainder of the paper is structured as follows. Section 2 reviews some basic concepts. Section 3 establishes the basic theory of SCC-recursiveness in FAF. Section 4 demonstrates the SCC-recursive characterization of fuzzy

[1] Following [5], we adopt the term 'admissible semantics' for the convenience of presentation, even though it is not considered as a semantics in Dung's original work.

extension semantics. Section 5 uses an example to illustrate the SCC-recursive schema. Section 6 discusses related work and concludes the paper.

## 2 Preliminaries

### 2.1 Fuzzy Set Theory

**Definition 1.** ([36]) *A* fuzzy set *is a pair* $(X, S)$ *in which* $X$ *is a nonempty set called the* universe *and* $S : X \to [0, 1]$ *is the associated* membership function. *For each* $x \in X$, $S(x)$ *is called the* grade *of membership of* $x$ *in* $X$.

For convenience, when the universe $X$ is fixed, a fuzzy set $(X, S)$ is identified by its membership function $S$, which can be represented by a set of pairs $(x, a)$ with $x \in X$ and $a \in [0, 1]$. We stipulate that all pairs $(x, 0)$ are omitted from $S$. For any $X' \subseteq X$, we denote by $S|_{X'}$ the *restriction* of $S$ to $X'$: for any $x \in X'$, $S|_{X'}(x) = S(x)$, and for any $x \notin X'$, $S|_{X'}(x) = 0$.

For instance, the following are fuzzy sets with universe $\{A, B, C\}$:

$$S_1 = \{(A, 0.5)\}, S_2 = \{(B, 0.8), (C, 1)\}, S_3 = \{(A, 0.8), (B, 0.8), (C, 1)\}.$$

Note that $S_1(A) = 0.5, S_1(B) = S_1(C) = S_2(A) = 0$, and in $S_3$ every element has a non-zero grade. Evidently, $S_2$ is the restriction of $S_3$ on $\{B, C\}$, i.e., $S_2 = S_3|_{\{B,C\}}$.

A *fuzzy point* is a fuzzy set containing a unique pair $(x, a)$. We may identify a fuzzy point by its pair. For example, $S_1$ is a fuzzy point and identified by $(A, 0.5)$.

Let $S_1$ and $S_2$ be two fuzzy sets. Say $S_1$ is a *subset* of $S_2$, denoted by $S_1 \subseteq S_2$, if for any $x \in X$, $S_1(x) \leq S_2(x)$. Conventionally, we write $(x, a) \in S$ if a fuzzy point $(x, a)$ is a subset of $S$. Moreover, we shall use the following notations:

- the *union* of $S_1$ and $S_2$: $S_1 \cup S_2 = \{(x, \max\{S_1(x), S_2(x)\}) \mid x \in X\}$;
- the *intersection* of $S_1$ and $S_2$: $S_1 \cap S_2 = \{(x, \min\{S_1(x), S_2(x)\}) \mid x \in X\}$.

In the above example, $S_1(x) \leq S_3(x)$ for each element $x$, thus fuzzy point $S_1$ is a subset of $S_3$, written as $(A, 0.5) \in S_3$. Similarly, it is easy to check: (i) $S_2 \subseteq S_3$; (ii) $S_2 \cup S_3 = \{(A, 0.8), (B, 0.8), (C, 1)\}$; (iii) $S_1 \cap S_3 = \{(A, 0.5)\}$.

### 2.2 Fuzzy Argumentation Frameworks

In this paper, we focus on *fuzzy argumentation framework* and its associated *fuzzy extension semantics* introduced in [34].

**Definition 2.** *A* fuzzy argumentation framework *(FAF) over a finite set of arguments* $Args$ *is a pair* $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ *in which* $\mathcal{A} : Args \to [0, 1]$ *and* $\mathcal{R} : Args \times Args \to [0, 1]$ *are total functions.*

In the definition, $\mathcal{A}$ and $\mathcal{R}$ are fuzzy sets of arguments and attacks. $\mathcal{A}$ can be denoted by pairs $(A, \mathcal{A}(A))$ and $\mathcal{R}$ can be denoted by pairs $\big((A, B), \mathcal{R}(A, B)\big)$ or simply $\big((A, B), \mathcal{R}_{AB}\big)$. Moreover, we denote by $Att(A)$ the set of all attackers of $A$, i.e., $Att(A) = \{B \in Args \mid \mathcal{R}(B, A) > 0\}$. For instance, we depict an FAF over $Args = \{A, B, C\}$ in Fig. 1, where

$$\mathcal{F} = \langle\{(A, 0.8), (B, 0.7), (C, 0.9)\}, \{\big((A, B), 0.8\big), \big((B, C), 0.9\big)\}\rangle.$$

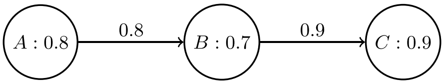



**Fig. 1.** A simple FAF

In the subsequent section, we apply a simple representation:

- $(A, a)$ can be represented as $A_a$,
- $\big((A, B), r\big)$ can be represented as $A \xrightarrow{r} B$.

Therefore, the above FAF can be represented as

$$\mathcal{F} = \langle\{A_{0.8}, B_{0.7}, C_{0.9}\}, \{A \xrightarrow{0.8} B, B \xrightarrow{0.9} C\}\rangle.$$

While arguments with conflict cannot be accepted together in classical semantics, semantics in quantitative settings allow for a certain degree of tolerance for internal conflicts among arguments [15,18,23], enabling weak attacks to be ignored. We review the notion of *tolerable* attacks from [34].

**Definition 3.** *Suppose* $(A, a)$ *attacks* $(B, b)$ *w.r.t.* $\mathcal{R}_{AB}$. *Then the attack is* tolerable *if* $a * \mathcal{R}_{AB} + b \leq 1$, *otherwise it is* sufficient. *Here,* $*$ *is a shorthand s.t.* $a * \mathcal{R}_{AB} = \min\{a, \mathcal{R}_{AB}\}$.

Note that the degrees of the attacker and the attack relation are aggregated toward the attacker. Intuitively, a tolerable attack can be *ignored* and a sufficient attack *weakens* the attackee.

**Definition 4.** *Let* $(A, a)$ *attacks* $(B, b)$ *w.r.t.* $\mathcal{R}_{AB}$. *Then* $(A, a)$ weakens $(B, b)$ *to* $(B, b')$ *where* $b' = \min\{1 - a * \mathcal{R}_{AB}, b\}$, *or precisely*

$$b' = \begin{cases} b & \text{if } a * \mathcal{R}_{AB} + b \leq 1, \\ 1 - a * \mathcal{R}_{AB} & \text{if } a * \mathcal{R}_{AB} + b > 1. \end{cases}$$

*We say that* $S$ weakens $(B, b)$ *to* $(B, b')$ *if* $\exists (C, c) \in S$ *weakens* $(B, b)$ *to* $(B, b')$.

The notion of *weakening defence* and its associated characteristic function are reviewed below, indicating that a fuzzy set of arguments defends a fuzzy argument by weakening its attackers.

**Definition 5.** *A fuzzy set $S$* weakening defends *a fuzzy argument $(A, a)$ iff, for any $(B, b)$ that sufficiently attacks $(A, a)$, $S$ weakens $(B, b)$ to $(B, b')$ s.t. $(B, b')$ tolerably attacks $(A, a)$.*

**Definition 6.** *The characteristic function of an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ is a function $F_{\mathcal{F}}$ from the set of all the subsets of $\mathcal{A}$ to itself, such that $\forall S \subseteq \mathcal{A}$, $F_{\mathcal{F}}(S) = \{(A, a) \in \mathcal{A} \mid S$ weakening defends $(A, a)\}$.*

The fuzzy extension semantics in [34] are reviewed as follows.

**Definition 7.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF and $E \subseteq \mathcal{A}$ a fuzzy set.*

*$E$ is* conflict-free *if all attacks in $E$ are tolerable. Suppose $E$ is conflict-free. Then we define:*

- *$E$ is an* admissible *fuzzy extension if $E$ weakening defends each element in $E$, i.e., $E \subseteq F_{\mathcal{F}}(E)$.*
- *$E$ is a* complete *fuzzy extension if it is admissible and contains all the elements in $\mathcal{A}$ that $E$ weakening defends, i.e., $E = F_{\mathcal{F}}(E)$.*
- *$E$ is a* preferred *fuzzy extension if it is a maximal admissible fuzzy extension.*
- *$E$ is the* grounded *fuzzy extension if it is the least fixed point of the characterization function $F_{\mathcal{F}}$.*

*The set of fuzzy extensions under a given semantics $\mathcal{S}$ is denoted by $\mathcal{E}_{\mathcal{S}}(\mathcal{F})$.*

As pointed out in [34], the stable and preferred fuzzy extensions coincide, so we omit the stable semantics. Intuitively, while classical extension semantics identify the arguments that can be accepted, fuzzy extension semantics quantify the degree to which arguments can be accepted—called the *acceptability degree*.

**Example 1.** *Continue considering the FAF depicted in Fig. 1. We compute a complete fuzzy extension $E$. Given that $A$ has no attackers, we have $E(A) = \mathcal{A}(A) = 0.8$. Since $B$ is weakened by $A$, it follows that $E(B) = 1 - E(A) * \mathcal{R}_{AB} = 0.2$. As $B$ is weakened, $C$ is weakening defended to the degree of 0.8, leading to $E(C) = 0.8$. Therefore, the acceptability degrees of $A$, $B$ and $C$ are 0.8, 0.2 and 0.8, respectively. Consequently, we obtain a complete fuzzy extension $\{(A, 0.8), (B, 0.2), (C, 0.8)\}$.*

## 3 SCC-Recursiveness in FAF

### 3.1 Graph Notations

The notion of SCC-recursiveness relies on the graph-theoretical notion of strongly connected components. To begin, we should integrate the graph notations into FAF.

**Definition 8.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF over $Args$. Given an argument $A \in Args$, we define the parent nodes of $A$ as $par_{\mathcal{F}}(A) = \{B \mid \mathcal{R}(B, A) > 0\}$. $A$ is called an* initial *node if $par_{\mathcal{F}}(A) = \varnothing$.*

**Definition 9.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF over Args, $A \in Args$ and $S, P \subseteq \mathcal{A}$. We define that:*

- *$S$ attacks $A$ iff $\exists B \in S$ s.t. $\mathcal{R}(B, A) > 0$;*
- *$A$ attacks $S$ iff $\exists B \in S$ s.t. $\mathcal{R}(A, B) > 0$;*
- *$S$ attacks $P$ iff $\exists A \in S$ and $\exists B \in P$ s.t. $\mathcal{R}(A, B) > 0$;*
- *$outpar_{\mathcal{F}}(S) = \{A \in Args \mid A \notin S \text{ and } A \text{ attacks } S\}$.*

The notions of *path* and *path-equivalence* are defined as follows.

**Definition 10.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF. We say that there is a* path *from $A_1$ to $A_n$ iff there is a sequence $\{A_1, A_2, ..., A_n\}$ such that $\mathcal{R}(A_i, A_{i+1}) > 0$ for $i \in \{1, ..., n-1\}$.*

**Definition 11.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF over Args. The binary relation of* path-equivalence *between nodes, denoted as $PE_{\mathcal{F}} \subseteq Args \times Args$, is defined as follows:*

- *$\forall A \in Args$, $(A, A) \in PE_{\mathcal{F}}$;*
- *given two distinct arguments $A, B \in Args$, $(A, B) \in PE_{\mathcal{F}}$ iff there is a path from $A$ to $B$ and a path from $B$ to $A$.*

The notion of *strongly connected components* is defined as follows.

**Definition 12.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF over Args. The equivalence classes under the path-equivalence relation are called* strongly connected components *(SCCs) of $\mathcal{F}$. We denote the set of SCCs of $\mathcal{F}$ by $SCCS_{\mathcal{F}}$. Given an argument $A \in Args$, the SCC that $A$ belongs to is denoted as $SCC_{\mathcal{F}}(A) = \{B \mid (A, B) \in PE_{\mathcal{F}}\}$.*

We extend the notion of parents to SCCs, representing the set of other SCCs that attack a given SCC $S$ as $sccpar_{\mathcal{F}}(S)$. Additionally, we introduce the concept of proper ancestors, denoted as $sccanc_{\mathcal{F}}(S)$.

**Definition 13.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF and $S \in SCCS_{\mathcal{F}}$. We define*

$$sccpar_{\mathcal{F}}(S) = \{P \in SCCS_{\mathcal{F}} \mid P \neq S \text{ and } P \text{ attacks } S\}$$
$$sccanc_{\mathcal{F}}(S) = sccpar_{\mathcal{F}}(S) \cup \bigcup_{P \in sccpar_{\mathcal{F}}(S)} sccanc_{\mathcal{F}}(P)$$

*An SCC $S$ is called* initial *if $sccpar_{\mathcal{F}}(S) = \varnothing$.*

For the purpose of decomposition, we introduce the notion of *restriction* of an FAF. Before the formal definition, consider attacks that are *always* tolerable. For instance, given an FAF $\mathcal{F} : A_{0.3} \xrightarrow{1.0} B_{0.2}$, it is evident that $\mathcal{A}(A) * R(A, B) + \mathcal{A}(B) = 0.3 * 1 + 0.2 \leq 1$, indicating that this attack is always tolerable in $\mathcal{F}$. We eliminate such always tolerable attacks when obtaining the restricted sub-frameworks from an FAF.

**Definition 14.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF over Args and $\mathcal{A}' \subseteq \mathcal{A}$ a fuzzy set. The* restriction *of $\mathcal{F}$ to $\mathcal{A}'$ is the sub-framework $\mathcal{F}\downarrow_{\mathcal{A}'} = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{R}'$ satisfies that*

- *if $\mathcal{A}'(A) * \mathcal{R}(A, B) + \mathcal{A}'(B) > 1$, then $\mathcal{R}'(A, B) = \mathcal{R}(A, B)$;*
- *otherwise $\mathcal{R}'(A, B) = 0$.*

For simplicity of discussion, we assume that the original FAF contains no always tolerable attacks in the subsequent discussion.

### 3.2 Basic Theory

While the idea behind SCC-recursiveness is intuitive and natural, the initial formalization of its required notions may seem quite complex due to its recursive nature. Therefore, we clarify its basic idea in this section.[2] In the following, we consider a generic FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a semantics $\mathcal{S} \in \{$*admissible, complete, preferred, grounded*$\}$.

To start, let us treat SCCs as single nodes. Then any FAF can be viewed as a directed acyclic graph; that is, the attack relation induces a partial order among the SCCs. Furthermore, we showed in [33] that the acceptability degree of an argument depends on its ancestor nodes under semantics $\mathcal{S}$. This implies that semantics can be computed following the sequence of SCCs. We begin by computing semantics for an initial SCC $\hat{S}$. To achieve this, we examine the sub-framework over $\hat{S}$ by restricting $\mathcal{F}$ to $\mathcal{A}|_{\hat{S}}$. The semantics of this sub-framework are processed by a *base function*, denoted as $\mathcal{BF}_{\mathcal{S}}$, which is defined to return the set of all fuzzy extensions under semantics $\mathcal{S}$.

Now, we arrive at the crucial problem: how to compute semantics for a given SCC $S$ after computing its ancestor SCCs. Suppose $A \in S$ and $E \in \mathcal{E}_{\mathcal{S}}(\mathcal{F})$. Let $\max\limits_{B \in outpar_{\mathcal{F}}(S)} E(B) * \mathcal{R}_{BA} = \tilde{a}$. Then $A$ is weakened to the lesser of $1 - \tilde{a}$ or $\mathcal{A}(A)$. Following Definition 5, only $A$'s unweakened degree can influence its target arguments within $S$. This implies that we only need to consider a restricted sub-framework over $S$ where arguments with the unweakened degree, e.g., the lesser of $1 - \tilde{a}$ or $\mathcal{A}(A)$ for $A$. Note that the relevant attacks may be suppressed when obtaining the restricted sub-framework, leading to the recursive decomposition of SCCs. Furthermore, an argument can be accepted to some degree only if it can be defended to that degree. Therefore, it is necessary to identify the degree to which an argument can be defended by $E$ from outside $S$. Consequently, we can distinguish three components:

- *Weakened Component*, denoted as $W_{\mathcal{F}}(S, E)$, which represents the weakened degree of arguments in $S$ by the ancestor SCCs.
- *Restricted Component*, denoted as $R_{\mathcal{F}}(S, E)$, which represents the unweakened degree of arguments in $S$.

[2] Referring to the example provided in Sect. 5 helps in understanding the concepts in this section.

– *Defended Component*, denoted as $D_{\mathcal{F}}(S,E)$, a subset of $R_{\mathcal{F}}(S,E)$ that represents the degree to which an argument can be defended by $E$ from outside $S$.

**Definition 15.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF, $E \subseteq \mathcal{A}$ and $S \in SCCS_{\mathcal{F}}$. We define that*

$$\begin{aligned}
W_{\mathcal{F}}(S,E) =& \{(A,a) \mid A \in S, a = \max_{B \in outpar_{\mathcal{F}}(S)} E(B) * \mathcal{R}_{BA}\} \\
R_{\mathcal{F}}(S,E) =& \{(A,a) \mid A \in S, a = \min\{1 - W_{\mathcal{F}}(S,E)(A), \mathcal{A}(A)\}\} \\
D_{\mathcal{F}}(S,E) =& \{(A,a) \mid A \in S, \forall B \in outpar_{\mathcal{F}}(S), E \text{ weakens } (B, \mathcal{A}(B)) \text{ to} \\
& (B,b) \text{ that tolerably attacks } (A,a)\}.
\end{aligned}$$

From the above discussion, the computation of semantics over $S$ depends on the restricted sub-framework $\mathcal{F}\downarrow_{R_{\mathcal{F}}(S,E)}$ and $D_{\mathcal{F}}(S,E)$. For the purpose of computation, we define a *generic function*, denoted as $\mathcal{GF}$, where

– input: a (possibly restricted) FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $C$;
– output: a set of subsets of $C$.

We will use the notation $\mathcal{GF}(\mathcal{F}, C)$ for the generic function, which is defined as follows. If $\mathcal{F}$ consists of exactly one SCC, then $\mathcal{GF}(\mathcal{F}, C)$ coincides with a *base function* $\mathcal{BF}_{\mathcal{S}}(\mathcal{F}, C)$, which is defined to obtain the fuzzy extensions of $\mathcal{F}$ contained in $C$ under semantics $\mathcal{S}$.[3] On the other hand, if $\mathcal{F}$ can be decomposed into several SCCs, then $\mathcal{GF}(\mathcal{F}, C)$ is obtained by recursively applying $\mathcal{GF}$ to each SCC of $\mathcal{F}$. Formally, this means that for any $S \in SCCS_{\mathcal{F}}$, $E|_S \in \mathcal{GF}(\mathcal{F}\downarrow_{R_{\mathcal{F}}(S,E)}, C')$, where $C'$ represents the defended component. Note that $C'$ is determined by considering both the attacks coming from outside $\mathcal{F}$ (as $\mathcal{F}$ is possibly restricted) and those coming from other SCCs within $\mathcal{F}$, yielding $C' = C \cap D_{\mathcal{F}}(S,E)$.

We now formally introduce SCC-recursiveness as a principle for fuzzy extension semantics.

**Definition 16 (SCC-recursiveness).** *A given semantics $\mathcal{S}$ is* SCC-recursive *iff for any FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$, $\mathcal{E}_{\mathcal{S}}(\mathcal{F}) = \mathcal{GF}(\mathcal{F}, \mathcal{A})$, where for any $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and for any fuzzy set $C \subseteq \mathcal{A}$, the function $\mathcal{GF}(\mathcal{F}, C) \subseteq 2^{\mathcal{A}}$ is defined as follows: for any $E \subseteq \mathcal{A}$, $E \in \mathcal{GF}(\mathcal{F}, C)$ if and only if*

– *in case $|SCCS_{\mathcal{F}}| = 1$, $E \in \mathcal{BF}_{\mathcal{S}}(\mathcal{F}, C)$,*
– *otherwise, $\forall S \in SCCS_{\mathcal{F}}$, $E|_S \in \mathcal{GF}(\mathcal{F}\downarrow_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S,E) \cap C)$,*

*where $\mathcal{BF}_{\mathcal{S}}(\mathcal{F}, C)$ is a function, called* base function*, that, given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ s.t. $|SCCS_{\mathcal{F}}| = 1$ and a fuzzy set $C \subseteq \mathcal{A}$, gives a subset of $2^{\mathcal{A}}$.*

As noticed before, the generic function $\mathcal{GF}(\mathcal{F}, C)$ is recursively defined. The base of the recursion is given by the base function $\mathcal{BF}_{\mathcal{S}}(\mathcal{F}, C)$, which returns

[3] Note that this base function covers the previous case for computing the semantics of initial SCCs.

the results for $\mathcal{F}$ consisting of a single SCC. When $\mathcal{F}$ consists of more than one SCC, the recursive step involves a decomposition schema along its SCCs. Consequently, to show that a semantics is SCC-recursive, it suffices to identify its base function and demonstrate that it fits the decomposition schema.

The definition naturally provides a schema for computing SCC-recursive semantics. Consider a generic FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ over $Args$. First, for any initial SCC $S$, we have $R_{\mathcal{F}}(S, E) = D_{\mathcal{F}}(S, E) = \mathcal{A}|_S$. The restricted sub-framework over $S$ is $\mathcal{F} \downarrow_{\mathcal{A}|_S}$, which clearly consists of a unique SCC. Then the base function $\mathcal{BF}_{\mathcal{S}}(\mathcal{F} \downarrow_{\mathcal{A}|_S}, \mathcal{A}|_S)$ is invoked, returning the set of fuzzy extensions of $\mathcal{F} \downarrow_{\mathcal{A}|_S}$ under semantics $\mathcal{S}$. The results are then utilized to identify the restricted sub-frameworks in subsequent SCCs. This procedure is recursively invoked and can be summarized as follows:

1. A (possibly restricted) FAF is partitioned into its SCCs; they form a partial order induced by the attack relation.
2. The set of fuzzy extensions over each initial SCC is determined using a semantic-specific base function.
3. For each fuzzy extension determined at step 2, the restricted and defended components within subsequent SCCs are identified; then the associated restricted sub-framework is taken into account.
4. The steps 1–3 are applied recursively on the restricted FAF obtained at step 3.

# 4 SCC-Recursive Characterization of Semantics

## 4.1 Generalized Fuzzy Extension Semantics

In order to develop an SCC-recursive characterization of semantics, it is necessary to redefine fuzzy extension semantics with reference to a specific subset $C$. In the following, we consider a generic FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $C \subseteq \mathcal{A}$. The notion of admissible fuzzy extension in $C$ is defined as follows.

**Definition 17.** *A fuzzy set $E \subseteq \mathcal{A}$ is an* admissible fuzzy extension in $C$ *iff $E \subseteq C$ and $E \in \mathcal{AE}(\mathcal{F})$. The set of admissible fuzzy extensions in $C$ is denoted as $\mathcal{AE}(\mathcal{F}, C)$.*

We introduce the notions of complete and preferred fuzzy extensions in $C$.

**Definition 18.** *A fuzzy set $E$ is a* complete fuzzy extension in $C$ *iff $E \in \mathcal{AE}(\mathcal{F}, C)$, and it contains all the elements in $C$ that $E$ weakening defend. The set of complete fuzzy extensions in $C$ is denoted as $\mathcal{CE}(\mathcal{F}, C)$.*

**Definition 19.** *A* preferred fuzzy extension in $C$ *is a maximal element of $\mathcal{AE}(\mathcal{F}, C)$. The set of preferred fuzzy extensions in $C$ is denoted as $\mathcal{PE}(\mathcal{F}, C)$.*

The following proposition shows that preferred fuzzy extensions always exist for any FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and for any $C \subseteq \mathcal{A}$.

**Proposition 1.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $C \subseteq \mathcal{A}$, there is always a preferred fuzzy extension $E \in \mathcal{PE}(\mathcal{F}, C)$.*

Proposition 2 shows that a preferred fuzzy extension in $C$ is also complete in $C$.

**Proposition 2.** *A preferred fuzzy extension in $C$ is also complete in $C$.*

**Definition 20.** *The* characteristic function of $\mathcal{F}$ in $C$ *is defined as follows:*

- $F_{(\mathcal{F},C)} : 2^C \to 2^C$
- $F_{(\mathcal{F},C)}(S) = \{(A, a) \mid (A, a) \in C, (A, a)$ *is weakening defended by* $S\}$.

It is easy to see that $F_{(\mathcal{F},C)}$ is monotonic w.r.t. fuzzy set inclusion. Then the grounded fuzzy extension in $C$ can be defined in terms of the least fixed point of the characteristic function in $C$.

**Definition 21.** *The* grounded fuzzy extension in $C$, *denoted as $GE(\mathcal{F}, C)$, is the least fixed point of $F_{(\mathcal{F},C)}$.*

The following proposition demonstrates that the grounded fuzzy extension always exists and is unique for any FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and any $C \subseteq \mathcal{A}$.

**Proposition 3.** *For any FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and any $C \subseteq \mathcal{A}$, $GE(\mathcal{F}, C)$ exists and is unique.*

Proposition 4 states that the grounded fuzzy extension in $C$ is also the least complete fuzzy extension in $C$.

**Proposition 4.** *$GE(\mathcal{F}, C)$ is the least complete fuzzy extension in $C$.*

Since the original version of fuzzy extension semantics is recovered by letting $C = \mathcal{A}$, the generalized definition covers the original ones.

### 4.2 SCC-Recursiveness of Fuzzy Extension Semantics

We first establish that admissible semantics fit the decomposition schema along SCCs. The characterization serves as the foundation for analyzing other semantics. This is achieved by Theorem 1, which requires two preliminary lemmas.

**Lemma 1.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF and $E$ be an admissible fuzzy extension. Suppose $(A, a) \in F_{\mathcal{F}}(E)$, denoting $SCC_{\mathcal{F}}(A)$ as $S$, then it holds that:*

- *$(A, a) \in D_{\mathcal{F}}(S, E)$;*
- *$(A, a)$ is weakening defended by $E|_S$ in $\mathcal{F}\downarrow_{R_{\mathcal{F}}(S,E)}$.*

**Lemma 2.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$, let $E \subseteq \mathcal{A}$ be a fuzzy set s.t. $\forall S \in SCCS_{\mathcal{F}}$, $E|_S \in \mathcal{AE}(\mathcal{F}\downarrow_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S, E))$. Then for any $\hat{S} \in SCCS_{\mathcal{F}}$ and any $(A, a) \in D_{\mathcal{F}}(\hat{S}, E)$, if $(A, a)$ is weakening defended by $E|_{\hat{S}}$ in $\mathcal{F}\downarrow_{R_{\mathcal{F}}(\hat{S},E)}$, then $(A, a)$ is weakening defended by $E$ in $\mathcal{F}$.*

**Theorem 1.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $E \subseteq \mathcal{A}$, it holds that: $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{AE}(\mathcal{F}, C)$ if and only if $\forall S \in SCCS_{\mathcal{F}}$,*

$$E|_S \in \mathcal{AE}(\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S, E) \cap C).$$

The following theorem shows that complete semantics also fit the decomposition schema.

**Theorem 2.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $E \subseteq \mathcal{A}$, it holds that: $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{CE}(\mathcal{F}, C)$ if and only if $\forall S \in SCCS_{\mathcal{F}}$,*

$$E|_S \in \mathcal{CE}(\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S, E) \cap C).$$

Next, we demonstrate that preferred semantics also fit the decomposition schema, as shown by Theorem 3 based on the following lemma.

**Lemma 3.** *Let $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an FAF, $E \in \mathcal{AE}(\mathcal{F})$ and $S \in SCCS_{\mathcal{F}}$. Then for any $\hat{E} \subseteq \mathcal{A}$, if $\hat{E}$ satisfies the following conditions:*

- *$E|_S \subseteq \hat{E} \subseteq D_{\mathcal{F}}(S, E)$, and*
- *$\hat{E}$ is admissible in $\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)}$, i.e., $\hat{E} \in \mathcal{AE}(\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)})$,*

*then $E \cup \hat{E}$ is admissible in $\mathcal{F}$.*

**Theorem 3.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $E \subseteq \mathcal{A}$, it holds that: $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{PE}(\mathcal{F}, C)$ if and only if $\forall S \in SCCS_{\mathcal{F}}$,*

$$E|_S \in \mathcal{PE}(\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S, E) \cap C).$$

Finally, we prove that the grounded semantics also fit the decomposition schema, as shown by the following Theorem 4.

**Theorem 4.** *Given an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ and a fuzzy set $E \subseteq \mathcal{A}$, it holds that: $\forall C \subseteq \mathcal{A}$, $E = GE(\mathcal{F}, C)$ if and only if $\forall S \in SCCS_{\mathcal{F}}$,*

$$E|_S = GE(\mathcal{F}{\downarrow}_{R_{\mathcal{F}}(S,E)}, D_{\mathcal{F}}(S, E) \cap C).$$

On the basis of the above theorems, we characterize that admissible, complete, preferred, and grounded fuzzy extension semantics are SCC-recursive by identifying the base functions in the theorem below.

**Theorem 5 (SCC-recursive Characterization).** *The admissible, complete, preferred and grounded semantics are SCC-recursive, characterized by the following base functions:*

- *$\mathcal{BF}_{\mathcal{AD}}(\mathcal{F}, C) \equiv \mathcal{AE}(\mathcal{F}, C)$;*
- *$\mathcal{BF}_{\mathcal{CO}}(\mathcal{F}, C) \equiv \mathcal{CE}(\mathcal{F}, C)$;*
- *$\mathcal{BF}_{\mathcal{PR}}(\mathcal{F}, C) \equiv \mathcal{PE}(\mathcal{F}, C)$;*
- *$\mathcal{BF}_{\mathcal{GR}}(\mathcal{F}, C) \equiv \{GE(\mathcal{F}, C)\}$.*

## 5 Illustrating Example for SCC-Recursive Schema

In this section, we use an example to illustrate the process of computing semantics using the SCC-recursive schema. Each FAF is recursively decomposed into many reduced sub-frameworks along the SCCs, enabling the efficient computation of the semantics of the original FAF based on these reduced sub-frameworks.

**Example 2.** *Consider an FAF $\mathcal{F} = \langle \mathcal{A}, \mathcal{R} \rangle$ depicted in Fig. 2, where*

$$\mathcal{A} = \{A_{0.8}, B_{0.8}, C_{0.6}, D_{0.9}, E_{0.8}, F_{0.8}, G_{1.0}, H_{1.0}, I_{1.0}\}$$
$$\mathcal{R} = \{A \xrightarrow{1.0} B, B \xrightarrow{1.0} A, B \xrightarrow{1.0} C, C \xrightarrow{1.0} D, D \xrightarrow{1.0} E, E \xrightarrow{1.0} F, F \xrightarrow{1.0} C, C \xrightarrow{1.0} E,$$
$$D \xrightarrow{1.0} G, E \xrightarrow{1.0} I, G \xrightarrow{1.0} I, I \xrightarrow{1.0} G, G \xrightarrow{1.0} H, H \xrightarrow{1.0} G, I \xrightarrow{1.0} H\}$$

*In this example, we compute a preferred fuzzy extension $E$ of $\mathcal{F}$. First, $\mathcal{F}$ can be partitioned into three SCCs: $S_1 = \{A, B\}$, $S_2 = \{C, D, E, F\}$, $S_3 = \{G, H, I\}$. Subsequently, we compute $E$ following the sequence of these SCCs.*

*For the initial SCC $S_1 = \{A, B\}$, it is easy to see that*

$$D_{\mathcal{F}}(S_1, E) = R_{\mathcal{F}}(S_1, E) = \{(A, 0.8), (B, 0.8)\},$$

*yielding the first sub-framework by restricting $\mathcal{F}$ to $R_{\mathcal{F}}(S_1, E)$:*

$$\mathcal{F}_1 = \mathcal{F}\downarrow_{R_{\mathcal{F}}(S_1,E)} = \langle \{A_{0.8}, B_{0.8}\}, \{A \xrightarrow{1.0} B, B \xrightarrow{1.0} A\} \rangle.$$

*Since $|SCCS_{\mathcal{F}_1}| = 1$, the base function $\mathcal{BF}_{\mathcal{PR}}$ is invoked. There are many results for selection, which potentially lead to different decomposition. We choose $E|_{S_1} = \{(A, 0.2), (B, 0.8)\}$ for illustration.*

*Next, we consider the SCC $S_2 = \{C, D, E, F\}$. Given that $S_1$ attacks $S_2$ and $E|_{S_1} = \{(A, 0.2), (B, 0.8)\}$, we have*

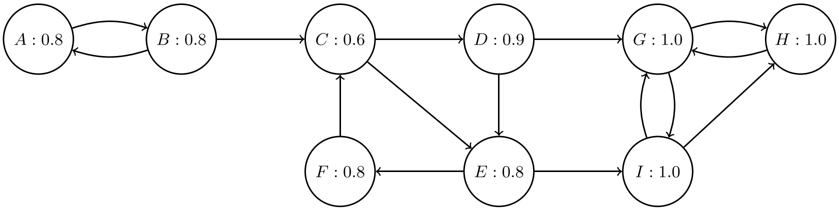
$$R_{\mathcal{F}}(S_2, E) = D_{\mathcal{F}}(S_2, E) = \{(C, 0.2), (D, 0.9), (E, 0.8), (F, 0.8)\}.$$

*Evidently, the attacks from $(C, 0.2)$ to $(E, 0.8)$ and $(F, 0.8)$ to $(C, 0.2)$ are always tolerable, and therefore according to Definition 14, the restriction of $\mathcal{F}$ to $R_{\mathcal{F}}(S_2, E)$ is*

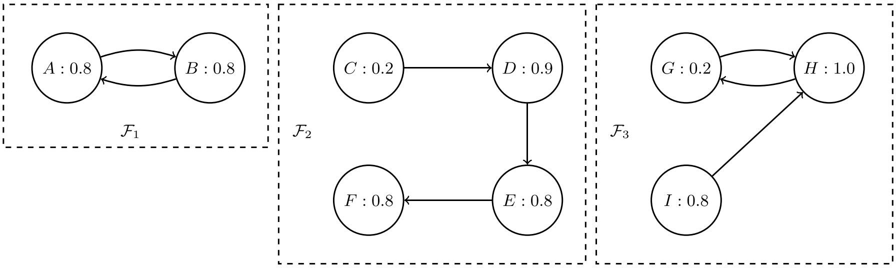
$$\mathcal{F}_2 = \mathcal{F}\downarrow_{R_{\mathcal{F}}(S_2,E)} = \langle \{C_{0.2}, D_{0.9}, E_{0.8}, F_{0.8}\}, \{C \xrightarrow{1.0} D, D \xrightarrow{1.0} E, E \xrightarrow{1.0} F\} \rangle.$$

*Then $\mathcal{F}_2$ can be recursively decomposed into four SCCs: $S_{2_1} = \{C\}$, $S_{2_2} = \{D\}$, $S_{2_3} = \{E\}$, $S_{2_4} = \{F\}$. It can be concluded that*

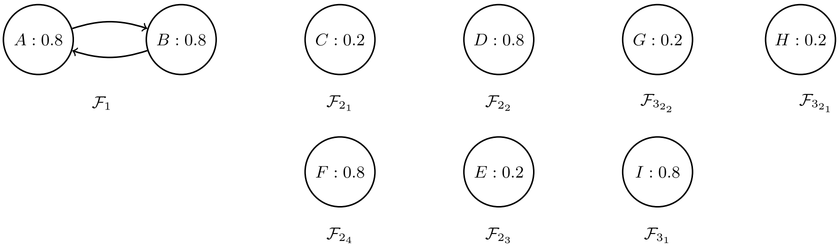
- *for $S_{2_1}$*
  - $R_{\mathcal{F}_2}(S_{2_1}, E|_{S_2}) = \{(C, 0.2)\}$;
  - $D_{\mathcal{F}_2}(S_{2_1}, E|_{S_2}) \cap D_{\mathcal{F}}(S_2, E) = \{(C, 0.2)\}$;
  - $\mathcal{F}_{2_1} = \mathcal{F}_2\downarrow_{\mathcal{R}_{\mathcal{F}_2}(S_{2_1}, E|_{S_2})} = \langle \{C_{0.2}\}, \varnothing \rangle$;
  - $E|_{S_{2_1}} = \{(C, 0.2)\}$.
- *for $S_{2_2}$*




(a) Before decomposition




(b) Intermediate decomposition




(c) After decomposition

**Fig. 2.** SCC-recursive decomposition in Example 2

- $R_{\mathcal{F}_2}(S_{2_2}, E|_{S_2}) = \{(D, 0.8)\}$;
- $D_{\mathcal{F}_2}(S_{2_2}, E|_{S_2}) \cap D_{\mathcal{F}}(S_2, E) = \{(D, 0.8)\}$;
- $\mathcal{F}_{2_2} = \mathcal{F}_2 \downarrow_{\mathcal{R}_{\mathcal{F}_2}(S_{2_2}, E|_{S_2})} = \langle \{D_{0.8}\}, \varnothing \rangle$;
- $E|_{S_{2_2}} = \{(D, 0.8)\}$.

– *for* $S_{2_3}$
- $R_{\mathcal{F}_2}(S_{2_3}, E|_{S_2}) = \{(E, 0.2)\}$;
- $D_{\mathcal{F}_2}(S_{2_3}, E|_{S_2}) \cap D_{\mathcal{F}}(S_2, E) = \{(E, 0.2)\}$;
- $\mathcal{F}_{2_3} = \mathcal{F}_2 \downarrow_{\mathcal{R}_{\mathcal{F}_2}(S_{2_3}, E|_{S_2})} = \langle \{E_{0.2}\}, \varnothing \rangle$;
- $E|_{S_{2_3}} = \{(E, 0.2)\}$.

– *for* $S_{2_4}$

- $R_{\mathcal{F}_2}(S_{2_4}, E|_{S_2}) = \{(F, 0.8)\}$;
- $D_{\mathcal{F}_2}(S_{2_4}, E|_{S_2}) \cap D_{\mathcal{F}}(S_2, E) = \{(F, 0.8)\}$;
- $\mathcal{F}_{2_4} = \mathcal{F}_2 \downarrow_{\mathcal{R}_{\mathcal{F}_2}(S_{2_4}, E|_{S_2})} = \langle\{F_{0.8}\}, \varnothing\rangle$;
- $E|_{S_{2_4}} = \{(F, 0.8)\}$.

*Consequently,* $E|_{S_2} = \{(C, 0.2), (D, 0.8), (E, 0.2), (F, 0.8)\}$.

*As far as the SCC* $S_3$ *is concerned, from that* $S_2$ *attack* $S_3$ *and* $E|_{S_2} = \{(C, 0.2), (D, 0.8), (E, 0.2), (F, 0.8)\}$, *we derive that*

$$R_{\mathcal{F}}(S_3, E) = D_{\mathcal{F}}(S_3, E) = \{(G, 0.2), (H, 1.0), (I, 0.8)\}.$$

*Since the attacks between* $(G, 0.2)$ *and* $(I, 0.8)$ *are always tolerable, according to Definition 14, the restriction of* $\mathcal{F}$ *to* $R_{\mathcal{F}}(S_3, E)$ *is*

$$\mathcal{F}_3 = \mathcal{F} \downarrow_{R_{\mathcal{F}}(S_3,E)} = \langle\{G_{0.2}, H_{1.0}, I_{0.8}\}, \{G \xrightarrow{1.0} H, H \xrightarrow{1.0} G, I \xrightarrow{1.0} H\}\rangle.$$

*Similarly,* $\mathcal{F}_3$ *can be recursively decomposed into two SCCs:* $S_{3_1} = \{I\}$, $S_{3_2} = \{G, H\}$.

– *For* $S_{3_1}$
  - $R_{\mathcal{F}_3}(S_{3_1}, E|_{S_3}) = \{(I, 0.8)\}$;
  - $D_{\mathcal{F}_3}(S_{3_1}, E|_{S_3}) \cap D_{\mathcal{F}}(S_3, E) = \{(I, 0.8)\}$;
  - $\mathcal{F}_{3_1} = \mathcal{F}_3 \downarrow_{\mathcal{R}_{\mathcal{F}_3}(S_{3_1}, E|_{S_3})} = \langle\{I_{0.8}\}, \varnothing\rangle$;
  - $E|_{S_{3_1}} = \{(I, 0.8)\}$.

*For* $S_{3_2}$, *since* $S_{3_1}$ *attacks* $S_{3_2}$ *and* $E|_{S_{3_1}} = \{(I, 0.8)\}$, *we obtain*

$$R_{\mathcal{F}_3}(S_{3_2}, E|_{S_3}) = D_{\mathcal{F}_3}(S_{3_2}, E|_{S_3}) = \{(G, 0.2), (H, 0.2)\}.$$

*Clearly the attacks between* $(G, 0.2)$ *and* $(H, 0.2)$ *are always tolerable, therefore, the restriction of* $\mathcal{F}_3$ *to* $R_{\mathcal{F}}(S_{3_2}, E|_{S_3})$ *is*

$$\mathcal{F}_{3_2} = \mathcal{F}_3 \downarrow_{R_{\mathcal{F}_3}(S_{3_2}, E|_{S_3})} = \langle\{G_{0.2}, H_{0.2}\}, \varnothing\rangle.$$

*Then* $\mathcal{F}_{3_2}$ *can be recursively partitioned into* $S_{3_{2_1}} = \{H\}$ *and* $S_{3_{2_2}} = \{G\}$. *Similar to the above analysis, we derive that*

– *for* $S_{3_{2_1}}$
  - $R_{\mathcal{F}_{3_2}}(S_{3_{2_1}}, E|_{S_{3_2}}) = \{(H, 0.2)\}$;
  - $D_{\mathcal{F}_{3_2}}(S_{3_{2_1}}, E|_{S_{3_2}}) \cap D_{\mathcal{F}_3}(S_{3_1}, E|_{S_3}) \cap D_{\mathcal{F}}(S_3, E) = \{(H, 0.2)\}$;
  - $\mathcal{F}_{3_{2_1}} = \mathcal{F}_{3_2} \downarrow_{\mathcal{R}_{\mathcal{F}_{3_2}}(S_{3_{2_1}}, E|_{S_{3_2}})} = \langle\{H_{0.2}\}, \varnothing\rangle$;
  - $E|_{S_{3_{2_1}}} = \{(H, 0.2)\}$.

– *for* $S_{3_{2_2}}$
  - $R_{\mathcal{F}_{3_2}}(S_{3_{2_2}}, E|_{S_{3_2}}) = \{(G, 0.2)\}$;
  - $D_{\mathcal{F}_{3_2}}(S_{3_{2_2}}, E|_{S_{3_2}}) \cap D_{\mathcal{F}_3}(S_{3_1}, E|_{S_3}) \cap D_{\mathcal{F}}(S_3, E) = \{(G, 0.2)\}$;
  - $\mathcal{F}_{3_{2_2}} = \mathcal{F}_{3_2} \downarrow_{\mathcal{R}_{\mathcal{F}_{3_2}}(S_{3_{2_2}}, E|_{S_{3_2}})} = \langle\{G_{0.2}\}, \varnothing\rangle$;
  - $E|_{S_{3_{2_2}}} = \{(G, 0.2)\}$.

*Consequently,* $E|_{S_3} = \{(G, 0.2), (H, 0.2), (I, 0.8)\}$.

*As a result, the combination fuzzy extension*

$$E = \{(A, 0.2), (B, 0.8), (C, 0.2), (D, 0.8), (E, 0.2), (F, 0.8), (G, 0.2), (H, 0.2), (I, 0.8)\}$$

*is a preferred fuzzy extension of* $\mathcal{F}$.

## 6 Discussion and Conclusion

SCC-recursiveness was proposed in [7] as a powerful schema for characterizing semantics through the decomposition of AF along SCCs. This schema has been extensively studied in the literature.

First, it has proven useful in developing algorithms for solving semantics. Cerutti et al. designed an SCC-recursive algorithm for computing preferred semantics in [13] and further exploited the parallel computation in [14]. Baroni et al. proposed an incremental computation algorithm for solving semantics in the dynamics of AF based on the schema [8].

Second, this schema has facilitated the exploration of new semantics. In [7], Baroni et al. proposed *CF2* and *AD2* semantics by incorporating the concept of *conflict-freeness* and *admissibility* with the SCC-recursive schema. In [19], Dvořák and Gaggl proposed *stage2* semantics by combining stage semantics with the schema.

Third, the feasibility of SCC-recursiveness for various semantics has attracted considerable attention. In [31], Villata et al. proposed the so-called *attack semantics* and defined an SCC-recursive schema for this semantics using *attack labelings*. In [16], Dauphin et al. demonstrated that the SCC-recursive schema is inapplicable to *weakly admissible*, *weakly complete* and *weakly grounded semantics*, whereas Dvořák et al. confirmed its applicability to *weakly preferred semantics* in [21].

Finally, the schema has also been extended to various qualitative frameworks. In [10], Baumann and Spanring investigated the schema in *unrestricted AF*. In [35], Yu et al. examined the schema in *abstract agent AF*. In [22], Gaggl et al. studied the schema in *abstract dialectical frameworks*. In [20], Dvořák et al. explored the schema in *AF with collective attacks*.

Despite these substantial contributions, the exploration of SCC-recursiveness in quantitative frameworks remains relatively underdeveloped. Although Rienstra et al. [28] explored *SCC-decomposability* in probabilistic argumentation, proposing a factorization scheme aligned with SCCs, the principle of SCC-recursiveness has not been formally addressed in that setting.

In this paper, we demonstrated that SCC-recursiveness can be applied to characterize fuzzy extension semantics in FAF. To achieve this, we tailored the existing SCC-recursive schema, enabling the characterization of fuzzy extension semantics—including *admissible*, *complete*, *grounded* and *preferred*—through the recursive decomposition of FAF along its SCCs. Our contributions are twofold. Theoretically, we showed that SCC-recursiveness provides an alternative approach to characterize fuzzy extension semantics, offering a deep understanding and better insight into these semantics. Practically, we provided a sound and complete algorithm for computing fuzzy extension semantics. As illustrated by the provided example, this algorithm naturally reduces computational efforts when dealing with a large number of SCCs.

Future research can take several directions. First, it is worth investigating the development of specific algorithms for computing fuzzy extension semantics based on the SCC-recursive schema. Second, utilizing the schema to explore new semantics in fuzzy AF offers significant potential. Third, investigating SCC-recursiveness in other quantitative settings, such as probabilistic AF [24], is a desirable endeavor.

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## Appendix

A full version including proofs can be found at https://arxiv.org/abs/2006.08880.

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